Find the first 4 **non-zero** terms of the Taylor series for $f(x) = \sec x$ about $x = -\frac{\pi}{3}$.

SCORE: _____ / 25 PTS

ALL ITEMS 2 POINTS UNLESS OTHERWISE NOTED

n	$f^{(n)}(x)$	$f^{(n)}(-\frac{\pi}{3})$
0	$\sec x$	2
1	$\sec x \tan x$	$2(-\sqrt{3}) = -2\sqrt{3}$
2	$\sec x \tan^2 x + \sec^3 x$	$2(-\sqrt{3})^2 + 2^3 = 14$
3	$\sec x \tan^3 x + 2\sec^3 x \tan x + 3\sec^3 x \tan x = \sec x \tan^3 x + 5\sec^3 x \tan x$	$2(-\sqrt{3})^3 + 5(2)^3(-\sqrt{3}) = -6\sqrt{3} - 40\sqrt{3} = -46\sqrt{3}$

$$2 - 2\sqrt{3}\left(x - \frac{-\pi}{3}\right) + \frac{14}{2!}\left(x - \frac{-\pi}{3}\right)^2 - \frac{46\sqrt{3}}{3!}\left(x - \frac{-\pi}{3}\right)^3 + \cdots$$

$$= 2 - 2\sqrt{3}\left(x + \frac{\pi}{3}\right) + 7\left(x + \frac{\pi}{3}\right)^2 - \frac{23\sqrt{3}}{3}\left(x + \frac{\pi}{3}\right)^3 + \cdots$$

1

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3n-2)!\,x^n}{(n!)^3}$. SCORE: _____ / 15 PTS

$$\underbrace{\lim_{n\to\infty} \left| \frac{(3n-2)!}{(n!)^3} \cdot \frac{((n+1)!)^3}{(3(n+1)-2)!} \right|}_{6} = \lim_{n\to\infty} \frac{(3n-2)!}{(3n+1)!} \left(\frac{(n+1)!}{n!} \right)^3$$

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)^3}{(3n+1)(3n)(3n-1)}}_{6} = \underbrace{\frac{1}{27}}_{3} = R$$

OR

$$\lim_{n\to\infty} \left| \frac{(3(n+1)-2)!\,x^{n+1}}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n-2)!\,x^n} \right| = \lim_{n\to\infty} \left| \frac{(3n+1)!}{(3n-2)!} \left(\frac{n!}{(n+1)!} \right)^3 x \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3n+1)(3n)(3n-1)}{(n+1)^3} \right| |x| < 1 \longrightarrow$$

$$27|x| < 1 \longrightarrow |x| < \frac{1}{27} = R$$

Consider the following statements. **SCORE: _____ / 5 PTS**

(i) If $\sum_{n=1}^{\infty} b_n$ converges, and $a_n < b_n$ for all integers $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges MISSING $0 < a_n$

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges CONTRAPOSITIVE OF ABSOLUTE CONVERGENCE TEST

Which of the statements above are true ? **Circle the correct answer below.** 5

[a] neither is true [b] only (i) is true [c] only (ii) is true [d] both are true

Let $f(x) = e^{-x^2} \cos 2x$.

SCORE: _____ / 25 PTS

[a] Find the first 3 **non-zero** terms of the Maclaurin series for $f(x)$.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \dots = \underbrace{1 - x^2 + \tfrac{1}{2}x^4}_{} - \dots \quad 5$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots = \underbrace{1 - 2x^2 + \tfrac{2}{3}x^4}_{} - \dots \quad 5$$

$$e^{-x^2}\cos 2x = 1 + (-2-1)x^2 + \left(\tfrac{2}{3} + 2 + \tfrac{1}{2}\right)x^4 + \dots$$

$$= \underbrace{1 - 3x^2 + \tfrac{19}{6}x^4}_{8} + \dots$$

[b] Find $f^{(4)}(0)$.

$$3 \quad \underbrace{\frac{f^{(4)}(0)}{4!}x^4 = \frac{19}{6}x^4}_{}$$

$$f^{(4)}(0) = \underbrace{\frac{19 \cdot 4!}{6}}_{2} = 19 \cdot 4 = \underbrace{76}_{2}$$

Find the interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{(2x-6)^n \cos n\pi}{n \ln n}$. $\cos n\pi = (-1)^n$ **SCORE: _____ / 35 PTS**

$$\underbrace{\lim_{n\to\infty} \left| \frac{(2x-6)^{n+1}(-1)^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{(2x-6)^n (-1)^n} \right|} = \underbrace{\lim_{n\to\infty} \left| \frac{n}{n+1} \frac{\ln n}{\ln(n+1)} \right| |2x-6|}$$

$$= 1 \cdot 1 \cdot |2x-6| = \underbrace{|2x-6| < 1}$$

NOTE: $\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$

$= \underbrace{\lim_{n\to\infty} \frac{n+1}{n}} = \underbrace{1}$

IF $-1 < 2x-6 < 1$

$\underbrace{\frac{5}{2} < x < \frac{7}{2}}$

IF $x = \frac{5}{2}$

$\sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{n \ln n} = \underbrace{\sum_{n=2}^{\infty} \frac{1}{n \ln n}}$

$\frac{1}{x \ln x}$ $\overbrace{> 0}^{1}$, $\overbrace{\text{IS CONTINUOUS AND}}^{1}$

1 $\overline{\text{DECREASING}}$ ON $[2, \infty)$

SINCE $\left(\frac{1}{x \ln x}\right)' = \underbrace{-(x\ln x)^{-2}(\ln x + 1)} \underbrace{< 0}$

ON $[2, \infty)$

$\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{N\to\infty} \underbrace{\ln|\ln x|}\Big|_2^N = \underbrace{\lim_{N\to\infty} \left(\ln|\ln N| - \ln|\ln 2|\right) = \infty}_{1}$ $\underbrace{\text{DIVERGES}}$

$\underbrace{\text{SO } \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ DIVERGES}}$ 1

IF $x = \frac{7}{2}$

$\underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}}$ $\overbrace{\text{CONVERGES BY ALTERNATING SERIES TEST}}$

SINCE $\underbrace{\frac{1}{n \ln n} \to 0}_{1}$ $\underbrace{\text{AND IS DECREASING}}_{1}$

INTERVAL $= \underbrace{\left(\frac{5}{2}, \frac{7}{2}\right]}$

ALL ITEMS 2 POINTS UNLESS OTHERWISE NOTED

AJ and CJ are trying to determine if $\sum_{n=1}^{\infty} a_n$ converges, where $a_n = f(n)$. **SCORE: _____ / 5 PTS**

They correctly check that f is positive and continuous at all integers $n \geq 1$, and that f' is negative at all integers $n \geq 1$.

They correctly determine that $\int_1^{\infty} f(x)\,dx$ diverges. AJ concludes that $\sum_{n=1}^{\infty} a_n$ diverges, but CJ thinks they cannot make that conclusion.

Who is right, and why ?

CJ – f NEEDS TO BE POSITIVE, CONTINUOUS + DECREASING FOR ALL REAL NUMBERS IN $[1, \infty)$ (NOT JUST INTEGERS) TO USE INTEGRAL TEST

1

4

Find the sum of the series $\frac{1}{e}-\frac{1}{2e^2}+\frac{1}{3e^3}-\frac{1}{4e^4}+\frac{1}{5e^5}-\cdots$.

SCORE: _____ / 10 PTS

$$= \frac{1}{e} - \frac{\left(\frac{1}{e}\right)^2}{2} + \frac{\left(\frac{1}{e}\right)^3}{3} - \frac{\left(\frac{1}{e}\right)^4}{4} + \frac{\left(\frac{1}{e}\right)^5}{5} - \cdots$$

$$= \ln\left(1+\frac{1}{e}\right)$$

3 3 3

$$= \ln\left(\frac{e+1}{e}\right)$$

$$= \ln(e+1) - \ln e$$

$$= \ln(e+1) - 1$$

1

Determine if the series $\sum_{n=2}^{\infty} \frac{(\cos n)\sqrt{n+1}}{(n-1)^2}$ converges. **SCORE: _____ / 30 PTS**

1 $\underbrace{0 \le}\ \underbrace{\left|\frac{(\cos n)\sqrt{n+1}}{(n-1)^2}\right|}\ <\ \underbrace{\frac{\sqrt{n+1}}{(n-1)^2}} := a_n$

ALL ITEMS 2½ POINTS UNLESS OTHERWISE NOTED

LET $b_n = \underbrace{\frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \underbrace{\lim_{n\to\infty} \frac{\sqrt{n+1}}{(n-1)^2} \cdot \frac{n^2}{\sqrt{n}}} = \underbrace{\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \left(\frac{n}{n-1}\right)^2} = \sqrt{1} \cdot 1^2 = \underbrace{1}\ \underbrace{\neq 0}$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ $\underbrace{\text{CONVERGES } (p = \tfrac{3}{2} > 1) \text{ BY } p\text{-SERIES TEST}}$ 4

$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{(n-1)^2}$ $\underbrace{\text{CONVERGES BY LIMIT COMPARISON TEST}}$

$\sum_{n=1}^{\infty} \left|\frac{(\cos n)\sqrt{n+1}}{(n-1)^2}\right|$ $\underbrace{\text{CONVERGES BY COMPARISON TEST}}$

$\sum_{n=1}^{\infty} \frac{(\cos n)\sqrt{n+1}}{(n-1)^2}$ $\underbrace{\text{CONVERGES BY ABSOLUTE CONVERGENCE TEST}}$